A note on precised Hardy inequalities on Carnot groups and Riemannian manifolds

Emmanuel Russ, Yannick Sire

A NOTE ON PRECISED HARDY INEQUALITIES ON CARNOT GROUPS AND RIEMANNIAN MANIFOLDS

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ABSTRACT. We prove non local Hardy inequalities on Carnot groups and Riemannian manifolds, relying on integral representations of fractional Sobolev norms.

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CONTENTS

1. INTRODUCTION

In the whole paper, when two quantities $A(f)$ and $B(f)$ depend on a function f ranging in some space V, the notation

$$A(f) \lesssim B(f) \ \forall f \in V$$

means that there exists $C > 0$ such that $A(f) \leq CB(f)$ for all $f \in V$. Moreover, the notation

$$A(f) \sim B(f) \ \forall f \in V$$

means that there exists $C > 0$ such that $C^{-1}B(f) \leq A(f) \leq CB(f)$ for all $f \in V$.

1.1. **Setting of the problem.** The simplest Hardy inequality on $\mathbb{R}^n$ asserts that, if $n \geq 3$,

$$\int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^2} dx \lesssim \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx = \|u\|_{\dot{H}^1(\mathbb{R}^n)}, \ \forall u \in \mathcal{D}(\mathbb{R}^n). \tag{1.1}$$

A non local version of (1.1) can be given, where the $\dot{H}^1$ norm in the right hand side is replaced by a $\dot{H}^s$ norm for $0 < s < \frac{n}{2}$ (see [BCG06]):

$$\int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^{2s}} dx \lesssim \|u\|^2_{\dot{H}^s(\mathbb{R}^n)}, \ \forall u \in \mathcal{D}(\mathbb{R}^n). \tag{1.2}$$

When $0 < s < 1$, it is well-known (see for instance [AF03]) that $\|u\|_{\dot{H}^s(\mathbb{R}^n)}$ can be represented by means of an integral quantity involving first order differences of u, and (1.2) can therefore be rewritten as

$$\int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^{2s}} dx \lesssim \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy. \tag{1.3}$$

These Hardy inequalities (*i.e.* the local and the non local version) were transposed to the framework of the Heisenberg group in [BCG06, BCX05]. More precisely, in the Heisenberg group $\mathcal{H}^d$ ($d \geq 1$), the following Hardy inequality was established in [BCX05]:

$$\int_{\mathcal{H}^d} \frac{u^2(x)}{\rho^2(x)} dx \lesssim \|\nabla_{\mathcal{H}} u\|_2^2, \ \forall u \in \mathcal{D}(\mathcal{H}^d), \tag{1.4}$$

where $\rho(x)$ denotes the distance of x to the origin and $\nabla_{\mathcal{H}}$ stands for the gradient associated to the vector fields $Z_1, ..., Z_{2d}$ (see [BCX05] and the notations therein). The non local version of (1.4), which was proven in [BCG06] (where it was derived from precised inequalities involving Besov norms) says that, for $0 < s < d + 1$,

$$\int_{\mathcal{H}^d} \frac{u^2(x)}{\rho^{2s}(x)} dx \lesssim \|u\|^2_{\dot{H}^s}, \ \forall u \in \mathcal{D}(\mathcal{H}^d). \tag{1.5}$$

When $0 < s < 1$, an integral representation for the fractional Sobolev homogeneous norm was proven in [CRTN01] (note that an analogous representation holds in any connected Lie group wth polynomial volume growth, and even in any unimodular Lie group if one works with the inhomogeneous version of this norm), so that (1.5) can be rewritten as

$$\int_{\mathcal{H}^d} \frac{u^2(x)}{\rho^{2s}(x)} dx \lesssim \iint_{\mathcal{H}^d \times \mathcal{H}^d} \frac{|u(x) - u(y)|^2}{\rho(y^{-1}x)^{2d+2+2s}} dxdy. \tag{1.6}$$

Hardy inequalities in *local* versions on more general Lie groups, namely Carnot groups, were obtained in [Kom06]. In the present paper, we establish non local versions of these Hardy inequalities on Carnot groups, in the spirit of (1.3) and (1.6). We also investigate the similar problem on Riemannian manifolds.

1.2. **The case of Lie groups.** We now describe the general framework for Lie groups. Let G be a unimodular connected Lie group endowed with the Haar measure. By "unimodular", we mean that the Haar measure is left and right-invariant. If we denote by $\mathcal{G}$ the Lie algebra of G, we consider a family

$$\mathbb{X} = \{X_1, ..., X_k\}$$

of left-invariant vector fields on G satisfying the Hörmander condition, i.e. $\mathcal{G}$ is the lie algebra generated by the $X_i's$. A standard metric on G , called the Carnot-Caratheodory metric, is naturally associated with $\mathbb{X}$ and is defined as follows: let $\ell : [0, 1] \to G$ be an absolutely continuous path. We say that ℓ is admissible if there exist measurable functions $a_1, ..., a_k : [0, 1] \to \mathbb{C}$ such that, for almost every $t \in [0, 1]$, one has

$$\ell'(t) = \sum_{i=1}^{k} a_i(t) X_i(\ell(t)).$$

If ℓ is admissible, its length is defined by

$$|\ell| = \int_0^1 \left(\sum_{i=1}^{k} |a_i(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

For all $x, y \in G$, define $d(x, y)$ as the infimum of the lengths of all admissible paths joining x to y (such a curve exists by the Hörmander condition). This distance is left-invariant. For short, we denote by $|x|$ the distance between e, the neutral element of the group and x, so that the distance from x to y is equal to $|y^{-1}x|$.

For all $r > 0$, denote by $B(x, r)$ the open ball in G with respect to the Carnot-Caratheodory distance and by $V(r)$ the Haar measure of any ball.

We denote

$$-\Delta_G = -\sum_{i=1}^{k} X_i^2$$

the sub-laplacian on G and $\nabla_G = (X_1, ..., X_k)$ the associated gradient.

The Lie group G is called a Carnot group if and only if G is simply connected and the Lie algebra of G admits a stratification, i.e. there exist linear subspaces $V_1, ..., V_k$ of $\mathcal{G}$ such that

$$\mathcal{G} = V_1 \oplus ... \oplus V_k$$

which

$$[V_1, V_i] = V_{i+1}$$

for $i = 1, ..., k-1$ and $[V_1, V_k] = 0$. By $[V_1, V_i]$ we mean the subspace of $\mathcal{G}$ generated by the elements $[X, Y]$ where $X \in V_1$ and $Y \in V_i$. Recall that the class of Carnot groups is a strict subclass of nilpotent groups. Moreover, if G is a Carnot group, there exists $n \in \mathbb{N}$, called the homogeneous dimension of G, such that, for all $r > 0$,

$$V(r) \sim r^n \tag{1.7}$$

(see [FS82]). The Heisenberg group $\mathcal{H}^d$ is a Carnot group and $n = 2d + 2$.

Let G be a Carnot group, denote by δ the Dirac distribution supported at the origin and let u be a solution of

$$-\Delta_G u = \delta.$$

Define $N(x) = u(x)^{\frac{1}{2-n}}$ for $x \neq 0$ and $N(0) = 0$. The function N is an homogeneous norm on N by [Fol75]. Kombe [Kom06] proved the following Hardy inequality on G: let $\alpha > 2 - n$ and $u \in \mathcal{D}(G \setminus \{0\})$, then the following holds

$$\left(\frac{n+\alpha-2}{2}\right)^2 \int_G u^2(x) \frac{|\nabla_G N(x)|^2}{|N(x)|^2} N^\alpha(x) dx \leq \int_G |\nabla_G u(x)|^2 N^\alpha(x) dx. \tag{1.8}$$

We prove the following non local version of (1.8):

Theorem 1.1. *Let G be a Carnot group with homogeneneous dimension $n \geq 3$. Then for all $\alpha > 2 - n$ and all $s \in (0,1)$,*

$$\int_G u^2(x) \left(\frac{|\nabla_G N|}{|N|}\right)^s N^\alpha dx \lesssim \iint_{G\times G} \frac{|u(x) - u(y)|^2}{|y^{-1}x|^{n+s}} N^\alpha(x)\, dx\, dy\; \forall u \in \mathcal{D}(G\backslash\{0\}). \tag{1.9}$$

1.3. **The case of Riemannian manifolds.** A general principle was developed in [Car97] to derive Hardy inequalities on Riemannian manifolds. Let us recall here an example of such an inequality. Let M be a Riemannian manifold, denote by n its dimension, by $d\mu$ its Riemannian measure, by d the exterior differentiation and by Δ the Laplace-Beltrami operator. For all $x \in M$ and all $r > 0$, let $B(x, r)$ be the open geodesic ball centered at x with radius r, and $V(x, r)$ its measure.
Assume that $\rho : M \to [0, +\infty)$ satisfies

$$|d\rho| \leq 1 \text{ on } M, \tag{1.10}$$

and

$$\Delta\rho \le -\frac{C}{\rho} \text{ in the distribution sense,} \tag{1.11}$$

where $C > 0$. Then, for all $\alpha < C - 1$ and all $u \in \mathcal{D}(M \setminus \rho^{-1}(0))$,

$$\left(\frac{C-1-\alpha}{2}\right)^2 \int_M \left(\frac{u}{\rho}\right)^2 (x)\rho^\alpha(x)dx \le \int_M |du(x)|^2 \rho^\alpha(x)dx. \tag{1.12}$$

Moreover, if the codimension of $\rho^{-1}(0)$ is greater than $2-\alpha$, (1.12) holds for all function $u \in \mathcal{D}(M)$ (see [Car97], Théorème 1.4 and Remarque 1.5) .

We provide here a non local version of (1.12). To state this result, we introduce some extra assumptions on M. The first one is a Faber-Krahn inequality on M. For any bounded open subset $\Omega \subset M$, denote by $\lambda_1^D(\Omega)$ the principal eigenvalue of $-\Delta$ on Ω under the Dirichlet boundary condition. If $p \ge n$, consider the following Faber-Krahn inequality: there exists $C > 0$ such that

$$\lambda_1^D(\Omega) \ge C\mu(\Omega)^{\frac{2}{p}} \text{ for all bounded subset } \Omega \subset M. \tag{1.13}$$

Let $\Lambda_p > 0$ be the greatest constant for which (1.13) is satisfied. In other words,

$$\Lambda_p = \inf \frac{\lambda_1^D(\Omega)}{\mu(\Omega)^{\frac{2}{p}}},$$

where the infimum is taken over all bounded subsets $\Omega \subset M$. The Faber-Krahn inequality (1.13) is satisfied in particular when an isoperimetric inequality holds on M: namely there exists $C > 0$ and $p \ge n$ such that, for all bounded smooth subset $\Omega \subset M$,

$$\sigma(\partial\Omega) \ge C\mu(\Omega)^{1-\frac{1}{p}}, \tag{1.14}$$

where $\sigma(\partial\Omega)$ denotes the surface measure of $\partial\Omega$. If M has nonnegative Ricci curvature, then (1.14) with $p = n$ and (1.13) with $p = n$ are equivalent. More generally, if M has Ricci curvature bounded from below by a constant, (1.13) with $p > 2n$ implies (1.14) with $\frac{p}{2}$ ([Car96], Proposition 3.1, see also [Cou92] when the injectivity radius of M is furthermore assumed to be bounded). Note that there exists a Riemannian manifold satisfying (1.13) for some $p \ge n$ but for which (1.14) does not hold for any $p \ge n$ ([Car96], Proposition 3.4).

It is a well-known fact that (1.13) implies a lower bound for the volume of geodesic balls in M. Namely ([Car96], Proposition 2.4), if (1.13) holds, then, for all $x \in M$ and all $r > 0$,

$$V(x,r) \geq \left(\frac{\Lambda_p}{2^{p+2}}\right)^{\frac{p}{2}} r^p. \tag{1.15}$$

We will also need another assumption on the volume growth of the balls in M. Say that M has the doubling property if and only if there exists $C > 0$ such that, for all $x \in M$ and all $r > 0$,

$$V(x,2r) \leq CV(x,r). \tag{D}$$

There is a wide class of manifolds on which (D) holds. First, it is true on Lie groups with polynomial volume growth (in particular on nilpotent Lie groups). Next, (D) is true if M has nonnegative Ricci curvature thanks to the Bishop comparison theorem (see [BC64]). Recall also that (D) remains valid if M is quasi-isometric to a manifold with nonnegative Ricci curvature, or is a cocompact covering manifold whose deck transformation group has polynomial growth, [CSC95]. Contrary to the doubling property, the nonnegativity of the Ricci curvature is not stable under quasi-isometry.

We prove the following theorem.

Theorem 1.2. *Let M be a complete non compact Riemannian manifold. Assume that* (1.13) *holds and that M has the doubling property. Assume also that $C > 0$ and $\rho : M \to [0, +\infty)$ are such that* (1.10) *and* (1.11) *hold. Then, if $\alpha < C - 1$ and $\rho^{-1}(0)$ has codimension greater than $2 - \alpha$, one has, for all $s \in (0, 1)$,*

(1.16)

$$\int_M u^2(x)\rho^{s(\alpha-2)}(x)dx \lesssim \iint_{M\times M} \frac{|u(y)-u(x)|^2}{d(x,y)^{p+s}}\rho^\alpha(x)dxdy \; \forall u \in \mathcal{D}(M\backslash\rho^{-1}(0)).$$

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need to introduce an operator L_{N^α} on $L^2(G)$. Let $L^2(G, N^\alpha)$ denote the L^2 space on G with respect to the measure $N^\alpha dx$ and $H^1(G, N^\alpha)$ the Sobolev space defined by

$$H^1(G, N^\alpha) := \left\{ f \in L^2(G, N^\alpha);\ X_i f \in L^2(G, N^\alpha)\ \forall 1 \leq i \leq k \right\}.$$

Define now the operator L_{N^α} on $L^2(G, N^\alpha)$ by

$$L_{N^\alpha} u := -N^{-\alpha} \sum_{i=1}^k X_i(N^\alpha X_i u).$$

The domain of L_{N^α} is given by

$$\mathcal{D}\left(L_{N^\alpha}\right) = \left\{ u \in L^2(G, N^\alpha);\ N^{-\alpha} X_i(N^\alpha X_i u) \in L^2(G, N^\alpha)\ \forall 1 \leq i \leq k \right\}.$$

One has, for all $u \in \mathcal{D}(L_{N^\alpha})$ and all $v \in H^1(G, N^\alpha)$,

$$\int_G L_{N^\alpha} u(x) v(x) N^\alpha(x) dx = \int_G \sum_{i=1}^k X_i u(x) X_i v(x) N^\alpha(x) dx.$$

The operator L_{N^α} is therefore clearly symmetric and nonnegative on $L^2(G, N^\alpha)$, and the spectral theorem allows to define the usual powers $(L_{N^\alpha})^\beta$ for any $\beta > 0$ by means of spectral theory.

By the definition of L_{N^α}, (1.8) means, in terms of operators in $L^2(G, N^\alpha)$, that, for some $\lambda > 0$,

$$L_{N^\alpha} \geq \lambda \mu, \tag{2.17}$$

where μ is the multiplication operator by $\frac{|\nabla_G N|}{|N|}$. Using a functional calculus argument (see [Dav80], p. 110) one deduces from (2.17) that, for any $s \in (0, 2)$,

$$(L_{N^\alpha})^{s/2} \geq \lambda^{s/2} \mu^{s/2}$$

which implies, thanks to the fact $(L_{N^\alpha})^{s/2} = ((L_{N^\alpha})^{s/4})^2$ and the symmetry of $(L_{N^\alpha})^{s/4}$ on $L^2(G, N^\alpha)$, that

$$\int_G |u(x)|^2 \mu(x)^s N^\alpha(x) dx \leq C \int_G \left|(L_{N^\alpha})^{s/4} u(x)\right|^2 N^\alpha(x) dx =$$

$$C \left\|(L_{N^\alpha})^{s/4} u\right\|^2_{L^2(G, N^\alpha)}.$$

The conclusion of Theorem 1.1 follows now from the estimate

$$\left\|(L_{N^\alpha})^{s/4} u\right\|^2_{L^2(G)} \leq C \iint_{G \times G} \frac{|u(x) - u(y)|^2}{|y^{-1}x|^{n+2s}} N^\alpha(x) \, dx \, dy,$$

which is exactly the estimate for $M = N^\alpha$ provided in Lemmata 3.2 and 3.3 in [RS10] (remember that (1.7) holds).

3. Proof of Theorem 1.2

The proof of Theorem 1.2 relies again on estimates for the powers of a suitable operator. Namely, define

$$L_{\rho^\alpha} u = \rho^{-\alpha} \mathrm{div}(\rho^\alpha \nabla u),$$

where ∇ is the gradient induced by the Riemannian metric and div is the divergence operator on M. As before, L_{ρ^α} is a nonnegative symmetric operator on $L^2(M, \rho^\alpha dx)$ and $L^\beta_{\rho^\alpha}$ is defined by spectral theory for all $\beta > 0$. If μ denotes the multiplication operator by ρ^{-2}, (1.12) means that $L_{\rho^\alpha}\geq c\mu$ in $L^2(M,\rho^\alpha dx)$. Spectral theory then yields $L_{\rho^\alpha}^{s/2}\geq c\mu^{s/2}$, which means that

$$\int_M u^2(x)\rho^{s(\alpha-2)}(x)dx\lesssim \left\Vert L_{\rho^\alpha}^{s/4}u\right\Vert_{L^2(M,\rho^\alpha dx)}^2,$$

and we are therefore left with the task of checking

$$\left\Vert L_{\rho^\alpha}^{s/4}u\right\Vert_{L^2(M,\rho^\alpha dx)}^2\lesssim \iint_{M\times M}\frac{\left|u(y)-u(x)\right|^2}{d(x,y)^{p+s}}\rho^\alpha(x)dxdy.$$

To that purpose, one first notices that

$$\left\Vert L_{\rho^\alpha}^{s/4}u\right\Vert_{L^2(M,\rho^\alpha dx)}^2\lesssim \int_0^{+\infty} t^{-1-s/2}\left\Vert t\,L_{\rho^\alpha}\left(\mathrm{I}+t\,L_{\rho^\alpha}\right)^{-1}u\right\Vert_{L^2(M,\rho^\alpha dx)}^2 dt$$

and it is therefore enough to show that

$$\int_0^{+\infty} t^{-1-s/2}\left\Vert t\,L_{\rho^\alpha}\left(\mathrm{I}+t\,L_{\rho^\alpha}\right)^{-1}u\right\Vert_{L^2(M,\rho^\alpha dx)}^2 dt$$

$$\lesssim \iint_{M\times M}\frac{\left|u(y)-u(x)\right|^2}{d(x,y)^{p+s}}\rho^\alpha(x)dxdy.$$

The proof follows the same lines as the one of Lemma 3.3 in [RS10] and we will therefore be sketchy, only indicating the main differences. Using (D), one can pick up a countable family x_j^t, $j\in\mathbb{N}$, such that the balls $B\left(x_j^t,\sqrt{t}\right)$ are pairwise disjoint and

$$M=\bigcup_{j\in\mathbb{N}} B\left(x_j^t,2\sqrt{t}\right). \tag{3.18}$$

By (D), there exist constants $\widetilde{C}>0$ and $\kappa>0$ such that for all $\theta>1$ and all $x\in G$, there are at most $\widetilde{C}\,\theta^{2\kappa}$ indexes j such that $|x^{-1}x_j^t|\leq\theta\sqrt{t}$.

For fixed j, one has

$$t\,L_{\rho^\alpha}\left(\mathrm{I}+t\,L_{\rho^\alpha}\right)^{-1}u=t\,L_{\rho^\alpha}\left(\mathrm{I}+t\,L_{\rho^\alpha}\right)^{-1}g^{j,t}$$

where, for all $x\in M$,

$$g^{j,t}(x):=u(x)-m^{j,t}$$

and $m^{j,t}$ is defined by

$$m^{j,t}:=\frac{1}{V\left(x_j^t,2\sqrt{t}\right)}\int_{B\left(x_j^t,2\sqrt{t}\right)}u(y)dy.$$

Note that, here, the mean value of u is computed with respect to the Riemannian measure on M. Since (3.18) holds, one clearly has

$$\begin{aligned}\left\|t\,L_{\rho^\alpha}\,(\mathrm{I}+t\,L_{\rho^\alpha})^{-1}u\right\|^2_{L^2(M,\rho^\alpha dx)} &\leq \sum_{j\in\mathbb{N}}\left\|t\,L_{\rho^\alpha}\,(\mathrm{I}+t\,L_{\rho^\alpha})^{-1}u\right\|^2_{L^2\left(B(x_j^t,2\sqrt{t}),\rho^\alpha dx\right)}\\ &= \sum_{j\in\mathbb{N}}\left\|t\,L_{\rho^\alpha}\,(\mathrm{I}+t\,L_{\rho^\alpha})^{-1}g^{j,t}\right\|^2_{L^2\left(B\left(x_j^t,2\sqrt{t}\right),\rho^\alpha dx\right)},\end{aligned}$$

and it is therefore enough to ckeck

$$\begin{aligned}(3.19)\qquad &\sum_{j\in\mathbb{N}}\left\|t\,L_{\rho^\alpha}\,(\mathrm{I}+t\,L_{\rho^\alpha})^{-1}g^{j,t}\right\|^2_{L^2\left(B\left(x_j^t,2\sqrt{t}\right),\rho^\alpha dx\right)}\\ &\qquad\lesssim \iint_{M\times M}\frac{|u(y)-u(x)|^2}{d(x,y)^{p+s}}\rho^\alpha(x)dxdy.\end{aligned}$$

As in [RS10], this is a consequence of L^2 off-diagonal estimates for L_{ρ^α} and upper estimates for the functions $g^{j,t}$. Let us recall the off-diagonal estimates for L_{ρ^α} for completeness:

Lemma 3.1. *There exists C with the following property: for all closed disjoint subsets $E,F\subset M$ with $d(E,F)=:d>0$, all function $f\in L^2(M,\rho^\alpha dx)$ supported in E and all $t>0$,*

$$\begin{aligned}\left\|(I+t\,L_{\rho^\alpha})^{-1}f\right\|_{L^2(F,\rho^\alpha dx)}+\left\|t\,L_{\rho^\alpha}(I+t\,L_{\rho^\alpha})^{-1}f\right\|_{L^2(F,\rho^\alpha dx)}\leq\\ 8\,e^{-C\frac{d}{\sqrt{t}}}\left\|f\right\|_{L^2(E,\rho^\alpha dx)}.\end{aligned}$$

The proof of Lemma 3.1 is analogous to the one of Lemma 3.1 in [RS10].

As far as estimates for $g^{j,t}$ are concerned, set, for all $k\geq 1$,

$$C_0^{j,t}=B\left(x_j^t,4\sqrt{t}\right)\quad\text{and}\quad C_k^{j,t}=B\left(x_j^t,2^{k+2}\sqrt{t}\right)\setminus B\left(x_j^t,2^{k+1}\sqrt{t}\right),$$

and $g_k^{j,t}:=g^{j,t}\,\mathbf{1}_{C_k^{j,t}}$, $k\geq 0$, where, for any subset $A\subset M$, $\mathbf{1}_A$ is the usual characteristic function of A. We then have:

Lemma 3.2. *There exists $\bar{C}>0$ such that, for all $t>0$ and all $j\in\mathbb{N}$:*

A.

$$\left\|g_0^{j,t}\right\|^2_{L^2(C_0^{j,t},\rho^\alpha dx)}\leq\frac{\bar{C}}{t^{p/2}}\int_{B\left(x_j^t,4\sqrt{t}\right)}\int_{B\left(x_j^t,4\sqrt{t}\right)}|u(x)-u(y)|^2\,\rho^\alpha(x)dx\,dy.$$

B. *For all $k\geq 1$,*

$$\begin{aligned}&\left\|g_k^{j,t}\right\|^2_{L^2(C_k^{j,t},\rho^\alpha dx)}\leq\\ &\frac{\bar{C}}{(2^k\sqrt{t})^p}\int_{x\in B(x_j^t,2^{k+2}\sqrt{t})}\int_{y\in B(x_j^t,2^{k+2}\sqrt{t})}|u(x)-u(y)|^2\,\rho^\alpha(x)dx\,dy.\end{aligned}$$

The proof of Lemma 3.2 is analogous to the one of Lemma 3.4 in [RS10], the only extra ingredient being the lower bound (1.15) applied with the balls $B(x_j^t, 2^{k+2}\sqrt{t})$. We then conclude the proof of (3.19) in the same way as for the conclusion of the proof of Lemma 3.3 in [RS10].

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